

AC CIRCUITS

For a resistance R , inductance L , and capacitance C in series with a voltage source $V = V_0 \exp(i\omega t)$ (here $i = \sqrt{-1}$), the current is given by $I = dq/dt$, where q satisfies

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = V.$$

Solutions are $q(t) = q_s + q_t$, $I(t) = I_s + I_t$, where the steady state is $I_s = i\omega q_s = V/Z$ in terms of the impedance $Z = R + i(\omega L - 1/\omega C)$ and $I_t = dq_t/dt$. For initial conditions $q(0) \equiv q_0 = \bar{q}_0 + q_s$, $I(0) \equiv I_0$, the transients can be of three types, depending on $\Delta = R^2 - 4L/C$:

(a) Overdamped, $\Delta > 0$

$$q_t = \frac{I_0 + \gamma_+ \bar{q}_0}{\gamma_+ - \gamma_-} \exp(-\gamma_- t) - \frac{I_0 + \gamma_- \bar{q}_0}{\gamma_+ - \gamma_-} \exp(-\gamma_+ t),$$

$$I_t = \frac{\gamma_+ (I_0 + \gamma_- \bar{q}_0)}{\gamma_+ - \gamma_-} \exp(-\gamma_+ t) - \frac{\gamma_- (I_0 + \gamma_+ \bar{q}_0)}{\gamma_+ - \gamma_-} \exp(-\gamma_- t),$$

where $\gamma_{\pm} = (R \pm \Delta^{1/2})/2L$;

(b) Critically damped, $\Delta = 0$

$$q_t = [\bar{q}_0 + (I_0 + \gamma_R \bar{q}_0)t] \exp(-\gamma_R t),$$

$$I_t = [I_0 - (I_0 + \gamma_R \bar{q}_0)\gamma_R t] \exp(-\gamma_R t),$$

where $\gamma_R = R/2L$;

(c) Underdamped, $\Delta < 0$

$$q_t = \left[\frac{\gamma_R \bar{q}_0 + I_0}{\omega_1} \sin \omega_1 t + \bar{q}_0 \cos \omega_1 t \right] \exp(-\gamma_R t),$$

$$I_t = \left[I_0 \cos \omega_1 t - \frac{(\omega_1^2 + \gamma_R^2) \bar{q}_0 + \gamma_R I_0}{\omega_1} \sin(\omega_1 t) \right] \exp(-\gamma_R t),$$

Here $\omega_1 = \omega_0(1 - R^2 C/4L)^{1/2}$, where $\omega_0 = (LC)^{-1/2}$ is the resonant frequency. At $\omega = \omega_0$, $Z = R$. The quality of the circuit is $Q = \omega_0 L/R$. Instability results when L, R, C are not all of the same sign.